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Greedy algorithm for functions with low mixed smoothness☆

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Abstract

In this note, we investigate the efficiency of the greedy algorithm for the classes of multivariate periodic functions with low mixed smoothness in L_q with regard to the wavelet-type basis. We find that the order of greedy approximation in the case of low smoothness is different for some range of parameters.

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1. Introduction and main results

Let $T^d = [0, 1]^d$ be the d-dimensional torus, and let $L_q := L_q([0, 1]^d)$, $1 \leqslant q < \infty$, be the Banach space of measurable functions $f(x) = f(x_1, \ldots, x_d)$, which is 1-periodic with respect to each variable. Its norm is defined by

$$\|f\|_q := \left(\int_{[0,1]^d} |f(x)|^q \, dx \right)^{1/q} .$$

The aim of this note is to investigate the efficiency of the greedy algorithm for the classes of multivariate periodic functions with low mixed smoothness in L_q. Denote by $\mathcal{D}$ the set of dyadic intervals of $[0, 1]$, each interval I in $\mathcal{D}$ being of the form $I = [j2^{-k}, (j + 1)2^{-k}]$,

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$k = 0, 1, 2, \ldots,\ j = 0, 1, \ldots, 2^k - 1$. Denote by $\mathcal{D}^d$ the set of all dyadic intervals of $[0,1]^d$, each $I \in \mathcal{D}^d$ being of the form $I = I_1 \times \cdots \times I_d$ with $I_1, \ldots, I_d \in \mathcal{D}$. Assume that a given system $\Phi = \{\phi_I\}_{I \in \mathcal{D}^d}$ of functions ϕ_I indexed by dyadic intervals can be enumerated in such a way that $\{\phi_{I^j}\}_{j=1}^\infty$ is a basis for L_q $(1 \leqslant q < \infty)$. Then we define the greedy algorithm $G_m^q(\cdot, \Phi)$ $(1 \leqslant q < \infty)$ as follows. Let

$$f = \sum_{j=1}^{\infty} c_{I^j}(f, \Phi)\phi_{I^j}, \quad c_I(f, q, \Phi) := \|c_I(f, \Phi)\phi_I\|_q.$$

Denote by Λ_m the set of m dyadic intervals such that

$$\min_{I \in \Lambda_m} c_I(f, q, \Phi) \geqslant \sup_{J \notin \Lambda_m} c_J(f, q, \Phi).$$

The set Λ_m may not be unique but if this happens we may take any of such sets. We define the greedy operator $G^q(\cdot, \Phi)$ by

$$G_m^q(f) := G_m^q(f, \Phi) := \sum_{I \in \Lambda_m} c_I(f, \Phi)\phi_I.$$

The operator $G_m^q(\cdot, \Phi)$ is a non-linear and discontinuous operator (see [1,9,10,13]).

Let us recall the definition of the best m-term approximation. Denote by $M_m(\Phi)$ the set of all linear combinations of the form

$$g = \sum_{I \in \Lambda_m} a_I \phi_I,$$

where Λ_m is a set of m dyadic intervals, a_I are real numbers. For a function class $F \subset L_q$, we consider the quantity

$$\sigma_m(F, \Phi)_q := \sup_{f \in F} \sigma_m(f, \Phi)_q := \sup_{f \in F} \inf_{g \in M_m(\Phi)} \|f - g\|_q.$$

We call the quantities $\sigma_m(f, \Phi)_q$ and $\sigma_m(F, \Phi)_q$ the best m-term approximation of f and F with regard to Φ, respectively (see [1,9,10,12,13]).

For $e \subset e_d := \{1, 2, \ldots, d\}, r > 0$, let $D^{r^e} f(x) = \left(\prod_{j \in e} \frac{\partial^r}{\partial x_j^r}\right) f(x)$ be the generalized derivative of f in the sense of Weyl (see [6,7]). Then the Sobolev classes MW_p^r of functions with mixed derivative are defined as follows:

$$MW_p^r := \left\{ f \in L_p([0,1]^d) \;\Big|\; \|f\|_{W_p^r} := \sum_{e \subset e_d} \|D^{r^e} f\|_p \leqslant 1 \right\}, \quad 1 \leqslant p < \infty.$$

Let $r > 0$, and let $l > r$ be a fixed positive integer. Then the Hölder–Nikolskii classes MH_p^r of functions with mixed difference are defined in the following way (see [6,7]):

$$MH_p^r := \left\{ f \in L_p([0,1]^d) \Big| \; \|f\|_{H_p^r} := \sum_{e \subset e_d} \sup_{t > 0} \prod_{j \in e} t_j^{-r} \cdot \|\Delta_{t^e}^{l^e} f\|_p \leqslant 1 \right\}, \quad 1 \leqslant p < \infty,$$

where $t=(t_1,\dots,t_d)>0$, (i.e., $t_j>0$, $j=1,\dots,d$), and

$$\Delta_{t^e}^{l^e} f(x) := \left(\prod_{j\in e} \Delta_{t_j,j}^{l}\right) f(x),$$

$$\Delta_{t_j,j}^{l} f(x) := \sum_{k=0}^{l} (-1)^{l-k}\binom{l}{k} f(x_1,\dots,x_j+kt_j,\dots,x_d).$$

Denote by O the set of all orthogonal bases on $[0,1]^d$. For the above classes and the anisotropic classes, Temlyakov proved that the orthogonal basis U^d formed from the integer traslates of the mixed dyadic scales of the tensor product multivariate Dirichlet kernel (or more generally, the wavelet-type basis Ψ^d, see the definition in Section 2) is optimal in the sense of order among all orthogonal systems for some range of parameters (see [9,10]). For example, for $1<p<\infty$, $r>(1/2-1/p)_+$, it was shown in [9] that:

$$\sigma_m(MW_p^r, O)_2 := \inf_{D\in O} \sigma_m(MW_p^r, D)_2 \asymp \sigma_m(MW_p^r, U^d)_2,$$

$$\sigma_m(MH_p^r, O)_2 := \inf_{D\in O} \sigma_m(MH_p^r, D)_2 \asymp \sigma_m(MH_p^r, U^d)_2.$$

Furthermore, Temlyakov proved that for all $1<q,p<\infty$, the orders of the best m-term approximations $\sigma_m(MW_p^r, U^d)_q$ and $\sigma_m(MH_p^r, U^d)_q$ can be achieved by the greedy algorithm $G^q(\cdot, U^d)$. For $1<p,q<\infty$,

$$r_1(p,q) := \begin{cases} \max(1/p, 1/2) - 1/q, & q \geqslant 2, \\ (\max(2/p, 2/q) - 1)/q, & q < 2, \end{cases}$$

$$r_2(p,q) := \begin{cases} (1/p - 1/q)_+, & q \geqslant 2, \\ (\max(2/p, 2/q) - 1)/q, & q < 2, \end{cases}$$

Temlyakov obtained the following results (see [9]):

$$\sigma_m(MW_p^r, U^d)_q \asymp \sup_{f\in MW_p^r} \|f - G_m^q(f, U^d)\|_q \asymp m^{-r}(\log_2 m)^{(d-1)r},$$
$$\text{if } r > r_1(p,q), \tag{1.1}$$

$$\sigma_m(MH_p^r, U^d)_q \asymp \sup_{f\in MH_p^r} \|f - G_m^q(f, U^d)\|_q \asymp m^{-r}(\log_2 m)^{(d-1)(r+1/2)},$$
$$\text{if } r > r_2(p,q), \tag{1.2}$$

where $a_+ := \max\{a, 0\}$; $A \asymp B$ means that $A \ll B$ and $B \ll A$; and $A \ll B$ means that there exists a positive constant c such that $A \leqslant cB$.

However, for the wavelet-type basis Ψ^d, the greedy algorithm $G_m^q(\cdot, \Psi^d)$ does not provide asymptotically optimal error for the best m-term approximation, since the following result holds (see [8,13]):

$$\sup_{f\in L_q} \|f - G_m^q(f, \Psi^d)\|_q \Big/ \sigma_m(f, \Psi^d)_q \asymp (\log_2 m)^{(d-1)|1/2-1/q|} \quad (1<q<\infty). \tag{1.3}$$

The above formula (1.3) shows that using the greedy algorithm $G_m^q(\cdot, \Psi^d)$ we lost near-best accuracy for some functions $f\in L_q$, $q\neq 2$, while (1.1), and (1.2) indicate that for all $1<q,p<\infty$ and big enough r, the orders of the best m-term approximations $\sigma_m(MW_p^r, \Psi^d)_q$ and

$\sigma_m(MH^r_p, \Psi^d)_q$ can be achieved by the greedy algorithm $G^q_m(\cdot, \Psi^d)$. How about the efficiency of the greedy algorithm for the classes $MW^r_p,\ MH^r_p$ without sufficiently large r? For the Sobolev classes MW^r_p, the case $1 < p \leqslant 2 \leqslant q < \infty$ has been studied in [9] for all $r > 1/p - 1/q$. In the case $1 < p < \infty$, $1 < q < 2$ we can extend the results of [9] to the case of low smoothness (the order is the same). The most interesting case is $2 < q, p < \infty$. Here we prove that the results from [9], concerning the greedy algorithm, cannot be extended in their form to the low smoothness case. We discover a new phenomenon: the order of greedy approximation in the case of low smoothness is different. This phenomenon is known in the case of Kolmogorov's widths (see [4,5]). For the Hölder–Nikolskii classes, we also obtain the upper estimates in the case of low smoothness. Our main results are the following.

Theorem 1. *Let* $1 < p, q < \infty$. *Then for* $(1/p - 1/q)_+ < r \leqslant 1/2 - 1/q,\ p > 2,\ q > 2$, *we have*

$$\sup_{f \in MW^r_p} \| f - G^q_m(f, \Psi^d)\|_q \asymp m^{-r} (\log_2 m)^{\frac{(d-1)r}{2(r+1/q)}},$$

and for $(1/p - 1/q)_+ < r \leqslant (\max(2/p, 2/q) - 1)/q,\ q < 2$, *we have*

$$\sup_{f \in MW^r_p} \| f - G^q_m(f, \Psi^d)\|_q \asymp m^{-r} (\log_2 m)^{(d-1)r}.$$

Theorem 2. *Let* $1 < p < \infty,\ 1 < q < 2$, *and* $r > (1/p-1/q)_+$. *Then for* $r < (2/p-1)/q,\ p < 2$, *we have*

$$\sup_{f \in MH^r_p} \| f - G^q_m(f, \Psi^d)\|_q \ll m^{-r} (\log_2 m)^{(d-1)(\frac{1}{pq(r+1/q)}+r)},$$

for $r = (2/p - 1)/q,\ p < 2$, *we have*

$$\sup_{f \in MH^r_p} \| f - G^q_m(f, \Psi^d)\|_q \ll m^{-r} (\log_2 m)^{(d-1)(r+1/2)} \log_2(\log_2 m),$$

and for $p > q,\ (2/p - 1)_+/q < r \leqslant (2/q - 1)/q$, *we have*

$$\sup_{f \in MH^r_p} \| f - G^q_m(f, \Psi^d)\|_q \ll m^{-r} (\log_2 m)^{(d-1)(r+1/2)}.$$

Remark 1.1. The lower estimate corresponding to the third upper estimate in Theorem 2 follows from [9] (see [9] or Lemma 2.1 in Section 2).

Remark 1.2. We do not know the exact orders of $\sigma_m(MW^r_p, \Psi^d)_q$ for $2 < p,\ q < \infty,\ (1/p - 1/q)_+ < r \leqslant 1/2 - 1/q$ and $\sigma_m(MH^r_p, \Psi^d)_q,\ \sup_{f \in MH^r_p} \| f - G^q_m(f, \Psi^d)\|_q$ for $1 < p,\ q < 2,\ (1/p - 1/q)_+ < r \leqslant (2/p - 1)/q$. It is an open problem.

Remark 1.3. For $1 < p, q < \infty,\ r > (1/p - 1/q)_+$, from the above upper estimates and the known lower estimates of the best m-term approximation (see [9] or Lemma 2.1 in Section 2), we get

$$1 \leqslant \frac{\sup_{f \in MW_p^r} \|f - G_m^q(f, \Psi^d)\|_q}{\sigma_m(MW_p^r, \Psi^d)_q} \ll \begin{cases} (\log_2 m)^{(d-1)\frac{r(1/2-r-1/q)}{r+1/q}}, & q > 2,\ r < 1/2 - 1/q, \\ 1, & q \leqslant 2 \text{ or } q > 2,\ r \geqslant (1/2 - 1/q)_+ \end{cases}$$

and

$$1 \leqslant \frac{\sup_{f \in MH_p^r} \|f - G_m^q(f, \Psi^d)\|_q}{\sigma_m(MH_p^r, \Psi^d)_q} \ll \begin{cases} (\log_2 m)^{(d-1)\frac{2-p-pqr}{2pq(r+1/q)}}, & p, q < 2,\ r < (2/p - 1)/q, \\ \log_2(\log_2 m), & p, q < 2,\ r = (2/p - 1)/q, \\ 1, & q \geqslant 2 \text{ or } q < 2,\ r > (2/p - 1)_+/q. \end{cases}$$

2. Wavelet-type system and lower estimates

First, we discuss the wavelet-type system $\Psi^d = \{\psi_I\}_{I \in \mathcal{D}^d}$. For any $s \in \mathbb{Z}^d$, $s \geqslant 0$ (i.e., $s_j \geqslant 0$, $j = 1, \ldots, d$), we write $\rho(s) := \{I = I_1 \times \cdots \times I_d \in \mathcal{D}^d \mid |I_j| = 2^{-s_j},\ j = 1, 2, \ldots, d\}$, where $|I_j|$ denotes the length of the interval I_j. It is easy to see that $\#\rho(s) = 2^{|s|}$, where $\#\rho(s)$ denotes the number of intervals in $\rho(s)$, $|s| := s_1 + \cdots + s_d$.

We suppose the system Ψ^d satisfies the following properties:

(1) Ψ^d is a basis of the space $L_p([0, 1]^d)$ $(1 < p < \infty)$, that is, for each $f \in L_p([0, 1]^d)$, f has a unique representation

$$f = \sum_{I \in \mathcal{D}^d} f_I \psi_I = \sum_{s \geqslant 0} \delta_s f; \quad \delta_s f := \sum_{I \in \rho(s)} f_I \psi_I,\ f_I := c_I(f, \Psi^d)$$

and the sum converges in L_p. Furthermore, the system Ψ^d is L_p-equivalent to the Haar system $\mathcal{H}^d$ $(1 < p < \infty)$. Let $H(t) = \begin{cases} 1, & t \in [0, 1/2), \\ -1, & t \in [1/2, 1), \\ 0, & \text{otherwise} \end{cases}$ $h_I(t) = 2^{n/2} H(2^n t - k)$, for a dyadic interval $I = [k2^{-n}, (k+1)2^{-n}] \in \mathcal{D}$ and $h_{[0,1]}(t) = 1$. For $I = I_1 \times \cdots \times I_d \in \mathcal{D}^d$, we set

$$h_I(x) = h_{I_1}(x_1) \cdots h_{I_d}(x_d),\ x = (x_1, \ldots, x_d).$$

We say that the system Ψ^d is L_p-equivalent to the Haar system $\mathcal{H}^d := \{h_I\}_{I \in \mathcal{D}^d}$ if for any finite set $\Lambda \subset \mathcal{D}^d$ and for any coefficients c_I, we have (see [2])

$$\left\| \sum_{I \in \Lambda} c_I \psi_I \right\|_p \asymp \left\| \sum_{I \in \Lambda} c_I h_I \right\|_p .$$

Applying the methods in [2] or [11], we know that the system Ψ^d satisfies the Littlewood–Paley inequalities:

$$\|f\|_p \asymp \left\| \left(\sum_{I\in\mathcal{D}^d} \left|f_I h_I\right|^2 \right)^{\frac{1}{2}} \right\|_p \asymp \left\| \left(\sum_{s\geqslant 0} \left|\delta_s f\right|^2 \right)^{\frac{1}{2}} \right\|_p . \tag{2.1}$$

(2) For any $I \in \mathcal{D}^d,\ 1 < q, p < \infty$, we have

$$\|\psi_I\|_2 \asymp 1, \quad \|\psi_I\|_p \asymp |I|^{1/p-1/2},\ \|\psi_I\|_p \asymp \|\psi\|_q \cdot |I|^{\frac{1}{p}-\frac{1}{q}}. \tag{2.2}$$

(3) For $f \in L_p([0,1]^d),\ 1 < p < \infty$, we have

$$\|\delta_s f\|_p^p = \left\| \sum_{I\in\rho(s)} f_I \psi_I \right\|_p^p \asymp \sum_{I\in\rho(s)} \|f_I \psi_I\|_p^p. \tag{2.3}$$

(4) For $1 < p < \infty,\ r > 0$, we have the following representation theorems of functions with mixed smoothness by Ψ^d:

$$\|f\|_{W_p^r} \asymp \left\| \left(\sum_{I\in\mathcal{D}^d} |I|^{-2r} \left|f_I h_I\right|^2 \right)^{\frac{1}{2}} \right\|_p \asymp \left\| \left(\sum_{s\geqslant 0} 2^{2r|s|} \left|\delta_s f\right|^2 \right)^{1/2} \right\|_p , \tag{2.4}$$

$$\|f\|_{H_p^r} \asymp \sup_{s\geqslant 0} 2^{r|s|} \|\delta_s f\|_p. \tag{2.5}$$

We say that Ψ^d is a wavelet-type basis, if the system Ψ^d satisfies the above conditions, From [9,3,12], we know that the basis U^d, the basis V formed from the integer translates of the mixed dyadic scales of the tensor product multivariate de la Valleé Poussin kernel, and the multivariate tensor product periodic wavelet basis W^d with ∞-regular univariate wavelet, all they are examples of wavelet-type bases. From [9] we know the following

Lemma 2.1. *Suppose that* $1 < p, q < \infty$, *and* $r > (1/p - 1/q)_+$. *Then*

$$\sigma_m(MW_p^r, \Psi^d)_q \geqslant m^{-r} (\log m)^{(d-1)r},$$

$$\sigma_m(MH_p^r, \Psi^d)_q \geqslant m^{-r} (\log m)^{(d-1)(r+1/2)}. \tag{2.6}$$

Lemma 2.2. *Suppose that* $2 < q < \infty$, *and* $(1/p - 1/q)_+ < r \leqslant 1/2 - 1/q$. *Then*

$$\sup_{f\in MW_p^r} \|f - G_m^q(f, \Psi^d)\|_q \geqslant m^{-r} (\log_2 m)^{\frac{(d-1)r}{2(r+1/q)}}.$$

Proof. Choose two positive integers J and l such that $m \asymp 2^J J^{d-1}$, $2^{J+l} \leqslant m$ and $2^l \asymp (\log_2 m)^{(d-1)}$. Choose $s_0 \in \mathbb{Z}^d$, $s_0 \geqslant 0$ such that $|s_0| = J + l$. Let $A := \{I : |I| = 2^{-J+l'}\}$, $B := \rho(s_0)$, where $2^{l'} \asymp (\log_2 m)^{(d-1)\frac{1/2-(r+1/q)}{r+1/q}}$. Then $\#B = 2^{J+l} \leqslant m$. Let $g_{Aq} = \sum_{I\in A} |I|^{1/2-1/q} \psi_I$,

$g_{Bq} = \sum_{I\in B} |I|^{1/2-1/q}\psi_I$. From (2.2) we know that there exists a constant $c > 0$ such that $f_0 - G_m^q(f_0, \Psi^d) = g_{Aq}$, where $f_0 = g_{Aq} + cg_{Bq}$. Then by (2.1) and (2.4) we have

$$\|g_{Aq}\|_q \asymp 2^{(J-l')(1/q-1/2)} \left\| \left(\sum_{\|s\|_1=J-l'} \sum_{I\in\rho(s)} |h_I|^2 \right)^{1/2} \right\|_q \asymp 2^{(J-l')/q} J^{(d-1)/2}$$

and

$$\|g_{Aq}\|_{W_p^r} \asymp 2^{(r+1/q)(J-l')} J^{\frac{d-1}{2}}.$$

Since

$$\|g_{Bq}\|_{W_p^r} \asymp 2^{(J+l)(1/q-1/2)} 2^{r(J+l)} \left(\sum_{I\in B} \|\psi_I\|_p^p \right)^{1/p} \asymp 2^{(J+l)(r+1/q)},$$

we obtain

$$\|f_0\|_{W_p^r} \asymp 2^{(r+1/q)(J-l')} J^{\frac{d-1}{2}} + 2^{(J+l)(r+1/q)} \asymp 2^{(r+1/q)(J-l')} J^{\frac{d-1}{2}}$$

and

$$\sup_{f\in MW_p^r} \|f - G_m^q(f)\|_q \geqslant \|g_{Aq}\|_q \big/ \|f_0\|_{W_p^r}$$
$$\asymp 2^{(J-l')/q} J^{\frac{d-1}{2}} 2^{-(r+1/q)(J-l')} J^{-\frac{d-1}{2}} \asymp 2^{-rJ} J^{(d-1)\frac{r(1/2-(r+1/q))}{r+1/q}}.$$

Lemma 2.2 is proved. □

Remark 2.1. In the proof of Lemma 2.2, it is essential that the basis is equivalent to the tensor product Haar basis. Here properties (2.2)–(2.5) (with exception of the middle part in (2.4)) would be not sufficient for the given estimate. Also the example used in this proof does not improve (2.6).

3. Upper estimates

For $f \in L_q([0,1]^d),\ 1 < q < \infty$, we have

$$f = \sum_{I\in\mathcal{D}^d} f_I\psi_I = \sum_{s\geqslant 0} \delta_s f, \quad G_m^q(f) = G_m^q(f, \Psi^d) = \sum_{I\in\Lambda_m} f_I\psi_I,$$

where Λ_m is a set of m dyadic intervals satisfying

$$V_q := V_q(f) := \min_{I\in\Lambda_m} \|f_I\psi_I\|_q \geqslant \sup_{J\notin\Lambda_m} \|f_J\psi_J\|_q. \tag{3.1}$$

For $s \in \mathbb{Z}^d,\ s\geqslant 0$, we set

$$\rho'(s) := \Lambda_m \cap \rho(s), \quad \rho''(s) := \rho(s) \setminus \rho'(s).$$

Choose a positive integer J such that

$$m \geqslant 2\#\Big(\cup_{|s|\leqslant J} \rho(s) \Big), \quad m \asymp 2^J J^{d-1}. \tag{3.2}$$

Then

$$\|f-G_m^q(f)\|_q = \left\|\sum_{s\geqslant 0}\sum_{I\in\rho''(s)} f_I\psi_I\right\|_q \ll \sum_{n\geqslant 0}\left\|\sum_{|s|=n}\sum_{I\in\rho''(s)} f_I\psi_I\right\|_q =: \sum_{n\geqslant 0} T_n. \quad (3.3)$$

Lemma 3.1. *Suppose* $1<p<\infty$. *Then for any* $f\in MW_p^r$, *we have*

$$\left(\sum_{s\geqslant 0} 2^{rp_l|s|}\|\delta_s f\|_p^{p_l}\right)^{1/p_l} \ll \|f\|_{W_p^r} \ll \left(\sum_{s\geqslant 0} 2^{rp_u|s|}\|\delta_s f\|_p^{p_u}\right)^{1/p_u}, \quad (3.4)$$

where $p_l := \max(2,p)$; $p_u := \min(2,p)$.

The proof of the case $r=0$ (where $\|f\|_{W_p^0}=\|f\|_p$) is given in [6], the proof of the case $r>0$ is similar, we omit it.

Lemma 3.2. *Suppose that* $f\in MW_t^r$, *and* $r\geqslant 1/t-1/q$. *Then we have*

$$V_q = V_q(f) \ll \begin{cases} 2^{-(r+1/q)J} J^{-\frac{d-1}{t}}, & 2\leqslant t\leqslant q<\infty,\\ 2^{-(r+1/q)J} J^{-\frac{d-1}{2}}, & 1<t\leqslant q<2. \end{cases} \quad (3.5)$$

Proof. For $t<q$, $f\in MW_t^r$, we have

$$\begin{aligned}\#(\rho'(s))\cdot V_q^t &\leqslant \sum_{I\in\rho'(s)}\|f_I\psi_I\|_q^t \ll \sum_{I\in\rho'(s)} 2^{(1/t-1/q)t|s|}\cdot\|f_I\psi_I\|_t^t\\ &\ll 2^{(1/t-1/q)t|s|}\|\delta_s f\|_t^t. \end{aligned} \quad (3.6)$$

Since

$$m\cdot V_q^t = \sum_s \#(\rho'(s))\cdot V_q^t \leqslant V_q^t\cdot\#(\cup_{|s|\leqslant J}\rho(s)) + V_q^t\cdot\#(\cup_{|s|>J}\rho'(s))$$

for $r\geqslant 1/t-1/q$, $t\geqslant 2$, by (3.2) and (3.6) we get

$$2^J J^{d-1}\cdot V_q^t \ll V_q^t\cdot\#(\cup_{|s|>J}\rho'(s)) \ll \sum_{|s|>J} 2^{(1/t-1/q)t|s|}\|\delta_s f\|_t^t. \quad (3.7)$$

For $r\geqslant 1/t-1/q$, $t\geqslant 2$, by (3.4) we have

$$2^J J^{d-1}\cdot V_q^t \ll 2^{-(r-1/t+1/q)J}\left(\sum_{|s|>J} 2^{rtn}\|\delta_s f\|_t^t\right) \ll 2^{-(r-1/t+1/q)J}.$$

And for $1<t\leqslant q<2$, by (3.7) and (3.4) we have

$$\begin{aligned}2^J J^{d-1}\cdot V_q^t &\ll \left(\sum_{|s|>J} 2^{-(r-1/t+1/q)|s|t\cdot\frac{2}{2-t}}\right)^{\frac{2-t}{2}}\cdot\left(\sum_{|s|>n} 2^{2r|s|}\|\delta_s f\|_t^2\right)^{t/2}\\ &\ll 2^{-(r-1/t+1/q)tJ} J^{(d-1)(2-t)/2}.\end{aligned}$$

Hence (3.5) holds. The proof of Lemma 3.2 is complete. □

Applying the same methods as in Lemma 3.2, we can get

Lemma 3.3. *Let* $1 < t, q < \infty,\ q < 2,\ f \in MH_t^r$, *and* $r > (1/t - 1/q)_+$. *Then we have*

$$V_q = V_q(f) \ll 2^{-(r+1/q)J}. \tag{3.8}$$

Remark 3.1. The case $1 < t \leqslant q < 2$ in Lemma 3.2 follows from [9, Lemma 3.1], Lemma 3.3 follows from [9, Lemma 2.1]. In general, the technique for upper estimates here is a refinement of the technique from [9].

Lemma 3.4. *Let* $1 < t \leqslant q < \infty$. *Then for* $f \in MW_t^r$, *we have*

$$T_n := \left\| \sum_{|s|=n} \sum_{I \in \rho''(s)} f_I \psi_I \right\|_q \ll 2^{-(r-1/t+1/q)tn/q} n^{(d-1)\frac{q-t}{2q}} V_q^{1-t/q}. \tag{3.9}$$

Proof. By (2.1) and (2.4) we get

$$T_n \asymp \left\| \left(\sum_{|s|=n} \sum_{I \in \rho''(s)} |f_I h_I|^2 \right)^{1/2} \right\|_q ;$$

$$\|f\|_{W_t^r} \asymp \left\| \left(\sum_{n \geqslant 0} 2^{2rn} \sum_{|s|=n} \sum_{I \in \rho(s)} |f_I h_I|^2 \right)^{1/2} \right\|_t . \tag{3.10}$$

Then, by (3.1) we have

$$\begin{aligned}
&\left(\sum_{|s|=n} \sum_{I \in \rho''(s)} |f_I h_I(x)|^2 \right)^{q/2} \\
&\quad \leqslant \left(\sum_{|s|=n} \sum_{I \in \rho''(s)} 2^{-2rtn/q} \cdot 2^{2rtn/q} |f_I h_I(x)|^2 \right)^{q/2} \\
&\quad \leqslant \left(\sum_{|s|=n} \sum_{I \in \rho''(s)} 2^{2rn} |f_I h_I(x)|^2 \right)^{t/2} \left(\sum_{|s|=n} 2^{-\frac{2rtn}{q-t}} \sum_{I \in \rho''(s)} |f_I h_I(x)|^2 \right)^{(q-t)/2} \\
&\quad \ll \left(\sum_{|s|=n} \sum_{I \in \rho''(s)} 2^{2rn} |f_I h_I(x)|^2 \right)^{t/2} \left(\sum_{|s|=n} 2^{-\frac{2rtn}{q-t}} 2^{-2(1/2-1/q)n} 2^n V_q^2 \right)^{(q-t)/2} \\
&\quad \ll 2^{-(r-1/t+1/q)tn} n^{(d-1)(q-t)/2} V_q^{q-t} \left(\sum_{|s|=n} \sum_{I \in \rho''(s)} 2^{2rn} |f_I h_I(x)|^2 \right)^{t/2}.
\end{aligned}$$

Using (3.10) we get (3.9). Lemma 3.4 is proved. □

Lemma 3.5. *Let* $1 < q,\ p \leqslant t \leqslant 2 < \infty$. *Then for* $f \in MH_p^r$, *we have*

$$T_n := \left\| \sum_{|s|=n} \sum_{I \in \rho''(s)} f_I \psi_I \right\|_q \ll 2^{-(rp-(t-p)/q)n/t} n^{(d-1)/t} V_q^{1-p/t}. \tag{3.11}$$

Proof. Using the similar methods as in Lemma 3.4, we get

$$\begin{aligned}
T_n &:= \left\| \sum_{|s|=n} \sum_{I \in \rho''(s)} f_I \psi_I \right\|_q \ll \left\| \sum_{|s|=n} \sum_{I \in \rho''(s)} f_I \psi_I \right\|_t \ll \left(\sum_{|s|=n} \left\| \sum_{I \in \rho''(s)} f_I \psi_I \right\|_t^t \right)^{1/t} \\
&\ll \left(\sum_{|s|=n} 2^{tn(1/q-1/t)} \sum_{I \in \rho''(s)} \| f_I \psi_I \|_q^t \right)^{1/t} \\
&\ll \left(\sum_{|s|=n} 2^{tn(1/q-1/t)} \sum_{I \in \rho''(s)} \| f_I \psi_I \|_q^p \cdot V_q^{t-p} \right)^{1/t} \\
&\ll \left(\sum_{|s|=n} 2^{tn(1/q-1/t)} 2^{pn(1/p-1/q)} \sum_{I \in \rho(s)} \| f_I \psi_I \|_p^p \right)^{1/t} V_q^{1-p/t} \\
&\ll 2^{\frac{(t-p)n}{tq}} \left(\sum_{|s|=n} \| \delta_s f \|_p^p \right)^{1/t} V_q^{1-p/t} \\
&\ll 2^{-(rp-(t-p)/q)n/t} n^{(d-1)/t} V_q^{1-p/t}. \qquad \square
\end{aligned}$$

Lemma 3.6. *Let* $2 < p \leqslant q < \infty$, *and* $f \in MW_p^r$. *Then for* $1/p - 1/q < r < 1/2 - 1/q$, *we have*

$$\| f - G_m^q(f) \|_q \ll 2^{-rJ} J^{(d-1)\frac{r(1/2-(r+1/q))}{r+1/q}}.$$

Proof. Let $l = [u(d-1)\log_2 J],\ u = \frac{1/2-(r+1/q)}{r+1/q}$. By (3.3) we have

$$\| f - G_m^q(f) \|_q \leqslant \sum_{n \leqslant J-l} T_n + \sum_{n > J-l} T_n =: T' + T''.$$

Using Lemma 3.2 with $1/t = r + 1/q,\ 2 < t < p$, we get

$$V_q \ll 2^{-(r+1/q)J} J^{-(d-1)(r+1/q)}. \tag{3.12}$$

Applying (3.9) with $t = 2$ and p, we get

$$\begin{aligned}
T' &:= \sum_{n \leqslant J-l} T_n \ll \sum_{n \leqslant J-l} 2^{-(r-1/2+1/q)2n/q} n^{(d-1)\frac{q-2}{2q}} V_q^{1-2/q} \\
&\ll 2^{-2(r-1/2+1/q)(J-l)/q} J^{(d-1)\frac{q-2}{2q}} \left(2^{-(r+1/q)J} J^{-(d-1)(r+1/q)} \right)^{1-2/q} \\
&\ll 2^{-rJ} J^{(d-1)\frac{r(1/2-(r+1/q))}{r+1/q}}
\end{aligned}$$

and

$$T'' := \sum_{n>J-l} T_n \ll \sum_{n>J-l} 2^{-(r-1/p+1/q)pn/q} n^{(d-1)\frac{q-p}{2q}} \left(2^{-(r+1/q)J} J^{-(d-1)(r+1/q)}\right)^{1-p/q}$$
$$\ll 2^{-rJ} 2^{(r-1/p+1/q)pl/q} J^{(d-1)(1/2-r-1/q)(1-p/q)} \ll 2^{-rJ} J^{(d-1)\frac{r(1/2-(r+1/q))}{r+1/q}}.$$

Lemma 3.6 is proved. □

Lemma 3.7. *Suppose that* $f \in MW^r_p$, $1 < p \leqslant q < 2$, $1/p - 1/q < r \leqslant (2/p-1)/q$ or $2 < p \leqslant q < \infty$, $r = 1/2 - 1/q$. *Then we have*

$$\| f - G^q_m(f)\|_q \ll 2^{-rJ}.$$

Proof. By (3.3) we have

$$\| f - G^q_m(f)\|_q \leqslant \sum_{n \leqslant J} T_n + \sum_{n>J} T_n =: T1 + T2.$$

Using Lemma 3.2 with $t = \min(p, 2)$, we get

$$V_q \ll 2^{-(r+1/q)J} J^{-(d-1)/2}.$$

Applying the Littlewood–Paley inequality (2.1), we obtain

$$T1 := \sum_{n \leqslant J} T_n \asymp \sum_{n \leqslant J} \left\| \left(\sum_{|s|=n} \sum_{I \in \rho''(s)} \left| f_I h_I \right|^2 \right)^{\frac{1}{2}} \right\|_q$$
$$\ll \sum_{n \leqslant J} \left\| \left(\sum_{|s|=n} \sum_{I \in \rho(s)} 2^{-2(1/2-1/q)n} V_q^2 \left| h_I \right|^2 \right)^{\frac{1}{2}} \right\|_q$$
$$\ll \sum_{n \leqslant J} 2^{n/q} n^{(d-1)/2} V_q \ll 2^{-rJ}. \tag{3.13}$$

By (3.9) and (3.5) (or (3.12)), we have

$$T2 := \sum_{n>J} T_n \ll \sum_{n>J} 2^{-(r-1/p+1/q)pn/q} n^{(d-1)\frac{q-p}{2q}} V_q^{1-p/q}$$
$$\ll 2^{-(r-1/p+1/q)pJ/q} J^{(d-1)\frac{q-p}{2q}} \left(2^{-(r+1/q)J} J^{-(d-1)/2}\right)^{1-p/q} \ll 2^{-rJ}.$$

Lemma 3.7 is proved. □

Lemma 3.8. *Suppose that* $1 < q < 2$, $f \in MH^r_p$, *and* $(1/p - 1/q)_+ < r < (2/p-1)/q$. *Then we have*

$$\| f - G^q_m(f)\|_q \ll 2^{-rJ} J^{\frac{d-1}{pq(r+1/q)}}.$$

Proof. Let $l = [u(d-1)\log_2 J]$, $u = \frac{1}{p(r+1/q)}$. By (3.3) we have

$$\| f - G^q_m(f)\|_q \leqslant \sum_{n \leqslant J+l} T_n + \sum_{n>J+l} T_n =: T' + T''.$$

Using Lemmas 3.3 and 3.5 with $t = 2$, we get

$$T' := \sum_{n \leqslant J+l} T_n \ll \sum_{n \leqslant J+l} 2^{-(rp-(2-p)/q)n/2} n^{(d-1)/2} V_q^{1-p/2}$$
$$\ll 2^{-rJ} J^{(d-1)/2} 2^{-l(rp-(2-p)/q)/2} \ll 2^{-rJ} J^{\frac{d-1}{pq(r+1/q)}}.$$

Using Lemma 3.5 with $t = \max(p, q)$, we obtain

$$T'' := \sum_{n > J+l} T_n \ll \sum_{n > J+l} 2^{-(rp-(t-p)/q)n/t} n^{(d-1)/t} V_q^{1-p/t} \ll 2^{-rJ} J^{\frac{d-1}{pq(r+1/q)}}.$$

Lemma 3.8 is proved. □

Lemma 3.9. *Suppose that* $1 < p, q < 2,\ f \in MH_p^r$, *and* $r = (2/p - 1)/q$. *Then we have*

$$\|f - G_m^q(f)\|_q \ll 2^{-rJ} J^{(d-1)/2} \log_2 J.$$

Proof. Let $l = [u(d-1)\log_2 J],\ u = q/2$. By (3.3) we have

$$\|f - G_m^q(f)\|_q \leqslant \sum_{n \leqslant J} T_n + \sum_{n > J} T_n =: T1 + T2 = T1 + \sum_{J < n \leqslant J+l} T_n + \sum_{n > J+l} T_n$$
$$=: T1 + T3 + T4. \tag{3.14}$$

Using (3.13) and Lemma 3.3 we get

$$T1 := \sum_{n \leqslant J} T_n \ll \sum_{n \leqslant J} 2^{n/q} n^{(d-1)/2} V_q \ll 2^{-rJ} J^{(d-1)/2}. \tag{3.15}$$

Using Lemma 3.5 with $t = \max(p, q)$, we obtain

$$T4 \ll \sum_{n > J+l} 2^{-(rp-(t-p)/q)n/t} n^{(d-1)/t} V_q^{1-p/t} \ll 2^{-rJ} J^{(d-1)/2}.$$

Using Lemma 3.5 with $t = 2$ again, we get

$$T3 \ll \sum_{J < n \leqslant J+l} 2^{-(rp-(2-p)/q)n/2} n^{(d-1)/2} V_q^{1-p/2} \ll l \cdot 2^{-rJ} J^{(d-1)/2}$$
$$\ll 2^{-rJ} J^{(d-1)/2} \log_2 J.$$

Lemma 3.9 is proved. □

Lemma 3.10. *Suppose that* $1 < q < 2,\ p > q,\ f \in MH_p^r$, *and* $(2/p-1)_+/q < r \leqslant (2/q-1)/q$. *Then we have*

$$\|f - G_m^q(f)\|_q \ll 2^{-rJ} J^{(d-1)/2}.$$

Proof. Applying (3.14) and (3.15), we get

$$\|f - G_m^q(f)\|_q \ll 2^{-rJ} J^{(d-1)/2} + T2.$$

Using Lemmas 3.3 and 3.5 with $t = 2$, we obtain

$$T2 \ll \sum_{n > J} 2^{-(rp-(2-p)/q)n/2} n^{(d-1)/2} V_q^{1-p/2} \ll 2^{-rJ} J^{(d-1)/2}.$$

Lemma 3.10 is proved. □

Proof of Theorems 1 and 2. The lower estimates of the quantities $\sup_{f\in MW_p^r}\|f-G_m^q(f,\Psi^d)\|_q$ are given in Lemmas 2.1 and 2.2; the upper estimates of the quantities $\sup_{f\in MW_p^r}\|f-G_m^q(f,\Psi^d)\|_q$ and $\sup_{f\in MH_p^r}\|f-G_m^q(f,\Psi^d)\|_q$ are given in Lemmas 3.6–3.10. Theorems 1 and 2 are proved. □

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References

[1] R.A. Devore, Nonlinear approximation, Acta Numer. (1998) 51–150.
[2] R.A. Devore, S.V. Konyagin, V.N. Temlyakov, Hyperbolic wavelet approximation, Constr. Approx. 14 (1998) 1–26.
[3] Z. Dinh, Continuous algorithms in n-term approximation and non-linear widths, J. Approx. Theory 102 (2000) 217–242.
[4] B.S. Kashin, Widths of Sobolev classes of small-order smoothness, Vestnik Moskow Univ. (Ser. Mat. Mekh. Astro.) 5 (1981) 50–54 (in Russian).
[5] E. Kulanin, Estimates of diameters of Sobolev classes with smoothness of small order, Vestnik Moscow Univ. (Ser. Mat. Mekh. Astro.) 2 (1983) 24–30.
[6] V.N. Temlyakov, Approximation of functions with bounded mixed derivative, Proc. Steklov Inst. Math. 178 (1989) 1–122.
[7] V.N. Temlykov, Approximation of Periodic Functions, Nova Science Publishers, Inc., New York, 1993.
[8] V.N. Temlyakov, Nonlinear m-term approximation with regard to the multivariate Haar system, East J. Approx. 4 (1) (1998) 87–106.
[9] V.N. Temlykov, Greedy algorithms with regard to multivariate systems with special structure, Constr. Approx. 16 (2000) 399–425.
[10] V.N. Temlykov, Universal basis and greedy algorithms, Constr. Approx. 18 (2002) 539–550.
[11] H. Wang, Representation and approximation of multivariate functions with mixed smoothness by hyperbolic wavelets, J. Math. Anal. Appl. 291 (2004) 698–715.
[12] H. Wang, Sun Yongsheng, Representation and m-term approximation for anisotropic classes, in: C.M. Nikolskii et al. (Eds.), Theory of Approximation of Function and Applications, Institute of Mathematics of Russian Academy, Moskov, 2003, pp. 250–268.
[13] P. Wojtaszczyk, Greedy algorithm for general bi-orthogonal systems, J. Approx. Theory 107 (2000) 293–314.